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Pólya’s Theorem with Zeros

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Abstract

Let $\mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$ and let $\Delta_n$ denote the standard $n$-simplex
$
\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum x_i = 1\}$. Pólya’s Theorem says that if a form
(homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on $\Delta_n$, then for sufficiently
large $N \in \mathbb{N}$, the coefficients of $(X_1 + \cdots + X_n)^N p$ are positive. In 2001, Powers
and Reznick established an explicit bound for the $N$ in Pólya’s Theorem. The
bound depends only on information about $p$, namely the degree and the size
of the coefficients of $p$, and the minimum value of $p$ on the simplex.

This thesis is part of an ongoing project, started by Powers and Reznick
in 2006, to understand exactly when Pólya’s Theorem holds if the condition
“positive on $\Delta_n$” is relaxed to “nonnegative on $\Delta_n$”, and to give bounds in
this case. In this thesis, we will show that if a form $p$ satisfies a relaxed
version of Pólya’s Theorem, then the set of zeros of $p$ is a union of faces of
the simplex. We characterize forms which satisfy a relaxed version of Pólya’s
Theorem and have zeros on vertices. Finally, we give a sufficient condition
for forms with zero set a union of two-dimensional faces of the simplex to
satisfy a relaxed version of Pólya’s Theorem, with a bound.
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Chapter 1

Introduction

Throughout this thesis we work in the real polynomial ring in \( n \) variables. Fix a positive integer \( n \), let \( \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n] \), and let \( \mathbb{R}^+[X] \) denote the polynomials in \( \mathbb{R}[X] \) with nonnegative coefficients. A \textit{form} is a homogeneous polynomial. We let \( \Delta_n \) denote the standard simplex,

\[
\Delta_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1 \}.
\]

\textbf{Pólya’s Theorem.} If a form \( p \in \mathbb{R}[X] \) is positive on \( \Delta_n \), then for sufficiently large \( N \in \mathbb{N} \), the coefficients of \( (X_1 + \ldots + X_n)^N p \) are positive.

Pólya’s theorem appeared in 1928 [9] (in German) and is also in \textit{Inequalities} by Hardy, Littlewood, and Pólya [7] (in English). In Hardy, Littlewood, and Pólya’s words: “The theorem gives a systematic process for deciding whether a given form \( F \) is strictly positive for positive \( x \). We multiply repeatedly by \( \sum x_i \) and, if the form is positive, we shall sooner or later obtain a form with positive coefficients.” [7]

In 2001, Powers and Reznick [10] established an explicit bound for the \( N \) in Pólya’s Theorem in terms of the degree and the size of the coefficients of the given form, and the minimum value of the form on the simplex.
Before giving this result, we establish some notation. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^N \), \( X^\alpha \) denotes the monomial \( X_1^{\alpha_1} \cdots X_n^{\alpha_n} \). Let \( |\alpha| \) denote \( \sum \alpha_i \) and if \( |\alpha| = d \), define \( c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!} \). Given \( p \in \mathbb{R}[X] \), a form of degree \( d \), say \( p(X) = \sum_{|\alpha| = d} a_\alpha X^\alpha \), let \( L(p) \) be the maximum of \( |a_\alpha/c(\alpha)| \).

**Theorem 1** (Powers, Reznick). Suppose \( p(X) \in \mathbb{R}[X] \) is homogeneous of degree \( d \) with \( p(X) > 0 \) on \( \Delta_n \). Let \( \lambda \) be the minimum of \( p(x) \) for \( x \in \Delta_n \). Then for \( N > \frac{d(d-1) L(p)}{2 \lambda} - d \) the coefficients of \( (X_1 + \cdots + X_n)^N p \) are positive.

We describe a few applications of Pólya’s Theorem and this bound. Pólya’s Theorem has been used in the study of copositive programming. Let \( \mathbb{S}^n \) denote the \( n \times n \) symmetric matrices over \( \mathbb{R} \) and define the copositive cone

\[
C_n = \{ M \in \mathbb{S}^n \mid Y^T M Y \geq 0 \text{ for all } Y \in \mathbb{R}_+^n \}.
\]

Copositive programming is optimization over \( C_n \). By Pólya’s Theorem, the truncated cones

\[
C^r_n := \{ M \in \mathbb{S}^n \mid (\sum_i x_i)^r X^T M X \}
\]

have non-negative coefficients and will converge to \( C_n \). Using linear programming, membership in \( C^r_n \) can be determined numerically. De Klerk and Pasechnik [3] use this fact, along with the bound for Pólya’s Theorem from Theorem 1, to give results on approximating the stability number of a graph.

Handelman [5, 6] has studied a related question, namely, for which pairs \((q, f)\) of polynomials does there exist \( N \in \mathbb{N} \) so that \( q^N f \) has nonnegative coefficients? (See also de Angelis and Tuncel [2].) Pólya’s Theorem and the
generalizations described in this thesis (without the bound) can be deduced from Handelman’s work.

More recently, Schweighofer [13] used Pólya’s Theorem to give an algorithmic proof of Schmüdgen’s Positivstellensatz, which says that if the basic closed semialgebraic set \( K = \{ g_1 \geq 0, \ldots, g_k \geq 0 \} \) is compact and \( f > 0 \) on \( K \), then \( f \) can be written as a finite sum of products of the \( g_i \)’s and squares in \( \mathbb{R}[X] \). This can be used to give an algorithm for optimization of polynomials on compact semialgebraic sets; see [15] for details. Using the bound from Theorem 1, Schweighofer obtained complexity bounds for Schmüdgen’s Positivstellensatz [16].

In 2006, Powers and Reznick [11] extended their results to non-negative polynomials allowed to have a certain type of zero at vertices of \( \Delta_n \).

**Definition 1.** Let \( p(X) \in \mathbb{R}[X] \) be homogeneous of degree \( d \) and suppose \( p(X) \geq 0 \) on \( \Delta_n \). Write \( v_1, \ldots, v_n \) for the vertices of \( \Delta_n \), i.e., \( v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1) \). Then \( p \) has a simple zero at the unit vertex \( v_i \) if the coefficient of \( X_i^d \) in \( p \) is zero, but the coefficient of \( X_i^{d-1}X_j \) is non-zero (and necessarily positive) for each \( j \neq i \).

In [11], Powers and Reznick show that if a form \( p \) is positive on \( \Delta_n \) except for simple zeros at \( v_i \)’s, then \( (X_1 + \ldots + X_n)^Np \in \mathbb{R}^+[X] \), for some \( N \in \mathbb{N} \). The bound on \( N \) in this case depends on the size of the coefficients of \( p \), the minimum of \( p \) away from the zeros, and some other constants determined by the coefficients of \( p \).

In 2005, Schweighofer [14] gave a “localized” version of Pólya’s Theorem that gives a condition which implies the conclusion of Pólya’s theorem (with “positive coefficients” replaced by “nonnegative coefficients”). The idea is to find a representation of \( f \), which depends on \( x \in \Delta_n \), and which implies the conclusion of Pólya’s Theorem for coefficients corresponding to \( X^\alpha \), where \( \frac{\alpha}{|\alpha|} \) is contained in a neighborhood around \( x \).
Proposition 1 (Schweighofer). Let $f \in \mathbb{R}[X]$. Suppose that for every $x \in \Delta_n$ there are $m \in \mathbb{N}$, forms $g_1, \ldots, g_m$ and $h_1, \ldots, h_m \in \mathbb{R}^+[X]$ such that
1) $f = g_1h_1 + \ldots + g_mh_m$, and
2) $g_i(x) > 0$ for all $i$.
Then there exists $N \in \mathbb{N}$ such that $(X_1 + \ldots + X_n)^Nf \in \mathbb{R}^+[X]$.

This thesis is part of an ongoing project, begun in [11], to understand exactly when Pólya’s Theorem holds if the condition “positive on $\Delta_n$” is relaxed to “nonnegative on $\Delta_n$”, and to give bounds in this case. The author, along with Powers and Reznick, began work on this project in [1]; most of this work is contained in Chapter 3 and 4. In this work, we give a computational version of Proposition 1, replacing neighborhoods of $x$ with closed subsets of $\Delta_n$, along with a bound on $N$. We then obtain Proposition 1 as a corollary. Using this computational version of Proposition 1, we characterize forms that are positive on $\Delta_n$, apart from zeros at $v_i$’s, and satisfy the conclusion of Pólyas Theorem (with “positive coefficients” replaced by “nonnegative coefficients”). This is a generalization of the main result from [11].

In this thesis, we continue work begun in [1]. We establish possible locations for zeros of forms that satisfy a relaxed version of Pólya’s Theorem. We include the work from [1] mentioned above. We then extend previous results to forms with zeros on two-dimensional faces, including a bound in this case. Finally, we include examples of forms that illustrate the work included in this thesis. Following is an outline of the thesis.

In Chapter 2, we give some preliminary notation and results. We show that if a form $p$ satisfies Pólya’s Theorem, then the zero set of $p$ must be a union of faces of the simplex. We also give a necessary, but not sufficient, condition for $p$ to satisfy Pólya’s Theorem.

Chapters 3 and 4 contain work from [1]. Given $p$ positive apart from zeros on the vertices $\Delta_n$, we characterize those for which there is an $N$ so that the coefficients of $(X_1 + \ldots + X_n)^Np$ are nonnegative, and give a bound on $N$. 
The proof uses our computational version of Proposition 1 to get a bound on a “corner piece” of $\Delta_n$ and then noting $p$ is positive on what remains of $\Delta_n$, we can apply the generalization of Theorem 1 to get a bound here. We then obtain the main result from [11] as a corollary.

In Chapter 5, we look at forms positive on $\Delta_n$ apart from zeros on two-dimensional faces. We consider subsets of $\Delta_n$ containing one-dimensional faces, two-dimension faces, and the rest of $\Delta_n$. In each case, we find an appropriate representation of $p$, apply our computational version of Proposition 1, and establish a bound.

Very recently, we learned of related work by Hoi-Nam Mok and Wing-Keung To [8]. The main theorem in [8] is a sufficient condition for a form non-negative on the simplex to satisfy Pólya’s Theorem, with a bound. This implies the main result in Chapter 5, however our bound is different. The proof in [8] is different from our proof.
Chapter 2

Preliminaries

Let $P_{n,d} (\Delta_n)$ denote the set of degree $d$ forms in $n$ variables which are non-negative on $\Delta_n$ and let $Po(n, d)$ be the degree $d$ forms in $n$ variables for which there exists an $N \in \mathbb{N}$ such that $(X_1 + \ldots + X_n)^N p \in \mathbb{R}^+[X]$. In other words, $Po(n, d)$ are the forms which satisfy the conclusion of Pólya’s theorem, with “positive coefficients” replaced by “nonnegative coefficients.”

For $I \subseteq \{1, \ldots, n\}$, let $F(I)$ denote the face of $\Delta_n$ containing the vertices $\{v_i \mid i \in I\}$, i.e.,

$$F(I) = \{(u_1, u_2, \ldots, u_n) \in \Delta_n \mid u_j = 0 \text{ for } j \notin I\}.$$ 

Note that $F(\emptyset) = \Delta_n$ and for $i \in \{1, \ldots, n\}$, $v_i = F(\{i\})$. The relative interior of the face $F(I)$ is the set $\{(u_1, \ldots, u_n) \in F(I) \mid u_i > 0 \text{ for } i \in I\}$. For $f(x) \in \mathbb{R}[X]$, we denote by $Z(f)$ the zeros of $f$, i.e., $Z(f) = \{u \in \mathbb{R}^n \mid f(u) = 0\}$.

Given $f = \sum a_\alpha X^\alpha \in \mathbb{R}[X]$ let

$$\Lambda^+(f) := \{\alpha \in \mathbb{N}^n \mid a_\alpha > 0\},$$

$$\Lambda^-(f) := \{\beta \in \mathbb{N}^n \mid a_\beta < 0\}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be $n$-tuples in $\mathbb{N}^n$, and let $I \subseteq \{1, \ldots, n\}$. Then we write $\beta \preceq_I \alpha$ if $\beta_i \leq \alpha_i$ for all $i \in I$, and $\beta \prec_I \alpha$ if
\( \beta \preceq_I \alpha \) and there exists some \( j \in I \) such that \( \beta_j < \alpha_j \). Note if \( I = \{i\} \), so that \( F(I) \) is a vertex, then for a form \( p \in \mathbb{R}[X] \), if \( \alpha \neq \beta \), \( \beta \preceq_I \alpha \) implies \( \beta \prec_I \alpha \).

In this section, we start with some observations about the possible location of zeros for a form \( p \in Po(n,d) \). These results can be found without proof in [11].

**Proposition 2.** Suppose \( p \in Po(n,d) \). If \( p(u) = 0 \) for \( u \) a point in the relative interior of a face of \( \Delta_n \), then \( p \) vanishes everywhere on the face.

**Proof.** For ease of exposition, we assume the face is \( F(\{1,\ldots,k\}) \), where \( 1 \leq k \leq n \). Then \( u = (u_1,\ldots,u_k,0,\ldots,0) \) where each \( u_i > 0 \). By assumption, there is an \( N \in \mathbb{N} \) such that \( (X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X] \). Let \( q = (X_1 + \cdots + X_n)^N p \), then \( q(u) = 0 \).

It is easy to see we can write \( p = p_1 + p_2 \) where \( p_1 \in \mathbb{R}[X_1,\ldots,X_k] \), and every monomial of \( p_2 \) contains at least one of \( \{X_{k+1},\ldots,X_n\} \), or \( p_2 \equiv 0 \). Note that \( p_2(u) = 0 \). Write \( (X_1 + \cdots + X_n)^N p_1 = \sum b_\gamma X_1^{\gamma_1} \cdots X_k^{\gamma_k} \), where \( b_\gamma \geq 0 \) for all \( \gamma \). Then

\[
q = (X_1 + \cdots + X_n)^N p_1 + (X_1 + \cdots + X_n)^N p_2,
\]

hence \( q(u) = \sum b_\gamma u_1^{\gamma_1} \cdots u_k^{\gamma_k} \). Since \( b_\gamma \geq 0 \) and \( u_1^{\gamma_1},\ldots,u_k^{\gamma_k} > 0 \), \( q(u) = 0 \) implies \( b_\gamma = 0 \) for all \( \gamma \). Hence \( p_1 \equiv 0 \), which gives \( p(w) = 0 \) for all \( w \) of the form \( (w_1,\ldots,w_k,0,\ldots,0) \), i.e, all points on the face. Thus \( p \) vanishes everywhere on the face. \( \square \)

**Corollary 1.** If \( p(u) = 0 \) for \( u \) a point in the interior of \( \Delta_n \), then \( p \equiv 0 \).

**Corollary 2.** The set \( Z(p) \cap \Delta_n \) is a union of faces of \( \Delta_n \).

The preceding lemma and corollaries show we need only focus our attention on zeros on faces of the simplex. However, the location of the zeros does not determine if \( p \in Po(n,d) \), as shown by the following example from [11].
Example 1. The following forms are non-negative on $\Delta_3$ with zeros only at vertices:

\[
\begin{align*}
f &= xz^3 + yz^3 + x^2y^2 - xyz, \\
g &= x^2y + y^2z + z^2x - xyz.
\end{align*}
\]

We will show $f \notin Po(3, 3)$, but $g \in Po(3, 3)$. We claim that the coefficient of $x^{N+1}yz^2$ in $(x+y+z)^Nf$ is always negative. There is no contribution from the coefficient of $(x+y+z)^Nxz^3$ or $(x+y+z)^Nyz^3$ because the power of $z$ is too large and there is no contribution from $(x+y+z)^N x^2y^2$ because the power of $y$ is too large. Hence the only contribution comes from $(x+y+z)^N (-xyz^2)$ and thus the coefficient will always be $-1$. On the other hand, it is easy to compute that $(x+y+z)^3 g$ has only positive coefficients. Thus the location of the zeros of $p \in P_{n,d}(\Delta_n)$ is not enough to determine whether $p$ is in $Po(n, d)$ or not.

Let $p \in \mathbb{R}[X]$. Then write $p = p^+ - p^-$ where

\[
p^+ = \sum_{\alpha \in \Lambda^+(p)} a_{\alpha} X^{\alpha} \quad \text{and} \quad p^- = \sum_{\beta \in \Lambda^-(p)} b_{\beta} X^{\beta},
\]

with $a_{\alpha}, b_{\beta} \in \mathbb{R}^+$. Note $p^+, p^- \in \mathbb{R}^+[X]$.

Hence, for any $N \in \mathbb{N}$,

\[
(X_1 + \cdots + X_n)^N = (X_1 + \cdots + X_n)^N (p^+ - p^-) = (X_1 + \cdots + X_n)^N p^+ - (X_1 + \cdots + X_n)^N p^- = (X_1 + \cdots + X_n)^N \sum_{\alpha \in \Lambda^+} a_{\alpha} X^{\alpha} - (X_1 + \cdots + X_n)^N \sum_{\beta \in \Lambda^-} b_{\beta} X^{\beta} = \sum_{|\gamma|=N+d} A_{\gamma} X^{\gamma} - \sum_{|\gamma|=N+d} B_{\gamma} X^{\gamma},
\]
where, from calculations given in [10], we have

\[ A_\gamma = \sum_{\alpha \in \Lambda^+(p), \alpha \leq \gamma} \frac{N!}{(\gamma_1 - \alpha_1)! \cdots (\gamma_n - \alpha_n)!} \cdot a_\alpha \quad (2.1) \]

\[ B_\gamma = \sum_{\beta \in \Lambda^-(p), \beta \leq \gamma} \frac{N!}{(\gamma_1 - \beta_1)! \cdots (\gamma_n - \beta_n)!} \cdot b_\beta \quad (2.2) \]

Definition 2. For \( I \subseteq \{1, \ldots, n\} \), let \( \bar{I} \) denote \( \{1, \ldots, n\} \setminus I \).

Proposition 3. Let \( p \in P_{n,d}(\Delta_n) \) and suppose \( p \in Po(n,d) \). Let \( I \subseteq \{1, \ldots, n\} \) and suppose \( Z(p) \) contains \( F(I) \). Let \( \Lambda^+ = \Lambda^+(p) \) and \( \Lambda^- = \Lambda^-(p) \). Then for every \( \beta \in \Lambda^- \) there exists an \( \alpha \in \Lambda^+ \) so that \( \alpha \preceq I \beta \).

Proof. Since \( p \in Po(n,d) \), there exists \( N \in \mathbb{N} \) such that \( (X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X] \). Suppose our assumption does not hold, i.e., there is a \( \beta = (\beta_1, \ldots, \beta_n) \in \Lambda^- \) such that for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda^+ \), \( \alpha \not\preceq \beta \), i.e., \( \beta \prec \alpha \). Then, for each \( \alpha \in \Lambda^+ \), there is some \( j \in \{1, \ldots, n\} \), \( j \notin I \), so that \( \alpha_j > \beta_j \).

Fix \( i \in I \) and for each positive integer \( N \geq 1 \), define \( \gamma = (\gamma_1, \ldots, \gamma_n) \) as follows:

\[ \gamma_j := \begin{cases} 
\beta_j, & \text{if } j \neq i \\
N + \beta_i, & \text{if } j = i 
\end{cases} \quad (2.3) \]

Clearly, \( |\gamma| = N + d \). For every \( \alpha \in \Lambda^+ \), since \( \alpha_j > \beta_j = \gamma_j \) for some \( j \neq i \), \( \alpha \not\preceq \gamma \) for any \( \alpha \in \Lambda^+ \). Hence, by (2.1) \( A_\gamma = 0 \). Additionally, from (2.3), we have \( \gamma_j \geq \beta_j \) for any \( j \in \{1, \ldots, n\} \). Hence, (2.3) and (2.2) imply \( B_\gamma > 0 \). Thus, for every positive integer \( N \geq 1 \) we have constructed a \( \gamma \) with \( |\gamma| = N + d \) so that the coefficient of \( X^\gamma \) in \( (X_1 + \cdots + X_n)^N p \) is negative, contradicting \( p \in Po(n,d) \).

Proposition 3 gives a necessary but not sufficient condition for \( p \in Po(n,d) \). This is demonstrated in the following example.
Example 2. Consider the following form $p$:

$$
p(x, y, z, w) = x^4 + y^4 + x^2(w - z)^2 = x^4 + y^4 + x^2z^2 + x^2w^2 - 2x^2zw
$$

Clearly, since $p$ is a sum of squares, this form is nonnegative on $\Delta_n$, hence $p \in P_{4,4}(\Delta_4)$. Also, $Z(p) \cap \Delta_4 = \{x = y = 0\} = F(\{3, 4\})$.

We have the following:

$$
\Lambda^+(f) = \{(4, 0, 0, 0), (0, 4, 0, 0), (2, 0, 2, 0), (2, 0, 0, 2)\}.
$$

$$
\Lambda^-(f) = \{(2, 0, 1, 1)\}.
$$

It is easy to see $p$ satisfies the conditions of Proposition 3, since

$$
(2, 0, 2, 0) \preceq_{\{1,2,3\}} (2, 0, 1, 1)
$$

$$
(2, 0, 0, 2) \preceq_{\{1,2,4\}} (2, 0, 1, 1)
$$

$$
(2, 0, 2, 0) \preceq_{\{1,2\}} (2, 0, 1, 1)
$$

We will show that $p$ is not in $Po(4, 4)$. Consider the $x^2z^{N+1}w^{N+1}$ term in $(x + y + z + w)^{2N}p$. Then for $\gamma = (2, 0, N + 1, N + 1)$, from (2.1) we have

$$
A_\gamma = \frac{2(2N + 4)!}{0!0!(N - 1)!(N + 1)!}.
$$

Likewise, from (2.2) we have

$$
B_\gamma = \frac{2(2N + 4)!}{0!0!N!N!}.
$$

Thus $B_\gamma > A_\gamma$ which implies $p$ is not in $Po(4, 4)$. 
Chapter 3

A localized Pólya’s Theorem

The work in this chapter, which is from [1], is a computational version of Proposition 1. Proposition 1 says that given \( f(X) \in \mathbb{R}[X] \), not necessarily homogeneous, if we can find certain types of representations of \( f \), which depend on \( x \in \Delta_n \), then there is an \( N \in \mathbb{N} \) so that the coefficient of \( X^\alpha \) in \( (\sum X_i)^N f \) is nonnegative whenever \( \frac{\alpha}{|\alpha|} \) is contained in a neighborhood around \( x \). Taking a finite subcover from these neighborhoods yields a global \( N \). Our version of this result replaces neighborhoods with finitely many closed subsets of \( \Delta_n \) covering \( \Delta_n \), which allows us to give an explicit bound for the exponent \( N \) needed.

We first give a localized version of Theorem 1.

**Lemma 1.** Suppose \( S \subseteq \Delta_n \) is nonempty and closed, and \( p \in \mathbb{R}[X] \) is homogeneous of degree \( d \) such that \( p(x) > 0 \) for all \( x \in S \). Let \( \lambda \) be the minimum of \( p \) on \( S \). Then for

\[
N > \frac{d(d - 1) L(p)}{2} \frac{\lambda}{\lambda} - d
\]

and \( \beta \in \mathbb{N}^n \) such that \( \frac{\beta}{|\beta|} \in S \), the coefficient of \( X^\beta \) in \( (X_1 + \ldots + X_n)^N p \) is nonnegative.
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Proof. The proof, which we give for completeness, is identical to the proof of Theorem 1 in [10]. We start with the technique of Pólya’s proof of his theorem. For a positive number \( t \), a non-negative integer \( m \), and \( x \in \mathbb{R} \), define

\[
(x)_t^m := x(x-t) \cdots (x-(m-1)t) = \prod_{i=0}^{m-1} (x-it).
\]

Note for later reference that

\[
(ty)_t^d = \prod_{i=0}^{d-1} (ty - (i-1)t) = t^d(y)_1^d,
\]

and if \( m > n \) are both integers, then \((n)_t^m = 0\), since one of the factors in the definition is zero. It follows immediately that in the special case that \( x = k/M \) and \( t = 1/M \), where \( M \) is a positive integer, we have

\[
\left(\frac{k}{M}\right)^m_1^1 = \frac{1}{M^m} \prod_{i=0}^{m-1} (k-i) = \begin{cases} \frac{1}{M^m} \frac{k!}{(k-m)!} = \frac{m!}{M^m} \binom{k}{m}, & \text{if } m \leq k; \\ 0, & \text{otherwise}. \end{cases}
\]

We fix \( p = \sum a_\alpha X^\alpha \) and suppose that \( p > 0 \) on \( S \subseteq \Delta_n \). We assume throughout that \( d = \deg p > 1 \); the \( d = 1 \) case is trivial. Following Pólya, we make the explicit computation:

\[
(X_1 + \cdots + X_n)^N p(X_1, \ldots, X_n) = \sum_{|\beta| = N} \frac{N!}{\beta_1! \cdots \beta_n!} X_1^{\beta_1} \cdots X_n^{\beta_n} \times \sum_{|\alpha| = d} a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}.
\]

For \(|\beta| = N + d\), denote the coefficient of \( X_1^{\beta_1} \cdots X_n^{\beta_n} \) in \((\sum X_i)^N p(X)\) by \( A_\beta \). Then

\[
A_\beta = \sum_{|\alpha| = d, \alpha \leq \beta} \frac{N!}{(\beta_1 - \alpha_1)! \cdots (\beta_n - \alpha_n)!} \cdot a_\alpha
\]

\[
= \frac{N!(N + d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha| = d, \alpha \leq \beta} a_\alpha \prod_{\ell=1}^{n} \frac{\beta_\ell!}{(\beta_\ell - \alpha_\ell)!}(N + d)^{\alpha_\ell}.
\]
We now express $A_\beta$ using the $(x)_m$ notation and (3.2):

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha|=d} a_\alpha \left( \frac{\beta_1}{N+d} \right)^{\alpha_1} \cdots \left( \frac{\beta_n}{N+d} \right)^{\alpha_n}$$

(3.3)

If $\alpha \not\leq \beta$, then the extra terms added in (3.3) are just 0. Still following Pólya, define

$$p_t(x_1, \ldots, x_n) := \sum_{|\alpha|=d} a_\alpha (x_1)^{\alpha_1} \cdots (x_n)^{\alpha_n}.$$ 

Clearly, $p_t \to p$ uniformly on $\Delta_n$ as $t \to 0$, hence for $t$ sufficiently small, $p_t$ is also positive on $S$. In view of the foregoing, this means that for $N$ sufficiently large, and all $\beta = (\beta_1 N+d, \ldots, \beta_n N+d) \in S$

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} p_{(N+d)-1}(\frac{\beta_1}{N+d}, \ldots, \frac{\beta_n}{N+d}) > 0.$$ 

(3.4)

We now extend Pólya’s work. Drop the constant factor in (3.4) and set $t = \frac{1}{N+d}$, $y_k = \frac{\beta_k}{N+d}$, and keep in mind that $\sum_k y_k = 1$. We have

$$p_t(y_1, \ldots, y_n) = p(y_1, \ldots, y_n) - \sum_{|\alpha|=d} a_\alpha (y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n}).$$

If $(y_1, \ldots, y_n) \in S$, $p(y_1, \ldots, y_n) \geq \lambda$, hence

$$p_t(y_1, \ldots, y_n) \geq \lambda - L \sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} |y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n}|$$

(3.5)

If $\alpha_k > \beta_k$, then $(y_k)^{\alpha_k} = 0$, so $y_k^{\alpha_k} \geq (y_k)^{\alpha_k} \geq 0$ for all $k$; hence we may drop the absolute value in (3.5)

By the Multinomial Theorem,

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \cdots y_n^{\alpha_n} = (y_1 + \cdots + y_n)^d = 1.$$ 

By the iterated Vandermonde-Chu identity [10],

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n} = (y_1 + \cdots + y_n)^d = \prod_{k=0}^{d-1} (1 - kt).$$

(3.6)
CHAPTER 3. A LOCALIZED PÓLYA’S THEOREM

By (3.5), we are done if we can show that

$$\lambda - L(1 - (1 - t) \cdots (1 - (d - 1)t)) > 0.$$  \hspace{1cm} (3.7)

Suppose now that

$$t = \frac{1}{N + d} < \frac{2}{d(d - 1)L}.$$  

It is easy to prove by induction that if $0 \leq w_j \leq 1$, then $\prod (1 - w_j) \geq 1 - \sum w_j$. Since $\lambda \leq p(1, 0, \ldots, 0) \leq L$ and $d \geq 2$, we have $t < \frac{1}{d - 1}$, hence

$$(1 - (1 - t) \cdots (1 - (d - 1)t)) < t(1 + 2 + \cdots + (d - 1)) = t \left( \frac{(d - 1)d}{2} \right) < \frac{\lambda}{L},$$

and we are done. \hfill \Box

We want to apply Lemma 1 in the case where we have a representation of $p \in \mathbb{R}[X]$ of the type in Proposition 1 for a closed subset $S$ of $\Delta_n$. In other words, we want to write $p = g_1 h_1 + \cdots + g_m h_m$ where $h_i \in \mathbb{R}[X]^+$ and $g_i(x) > 0$ for all $x \in S$, then apply Lemma 1 to the $g_i$. Our result will hold for a possibly smaller subset $T \subseteq S$ due to the fact that the exponents of $(\sum X_i)^N p$ are not the same as the exponents of $(\sum X_i)^N g_j$. For our specific application in Chapter 4, we will be able to take $T = S$.

**Proposition 4.** Given $p \in \mathbb{R}[X]$ (not necessarily homogeneous) and a nonempty closed set $S \subseteq \Delta_n$ and suppose there exist homogeneous $g_1, \ldots, g_m \in \mathbb{R}[X]$, and $h_1, \ldots, h_m \in \mathbb{R}[X]^+$ with

1. $p = g_1 h_1 + \cdots + g_m h_m$, and
2. $g_i(x) > 0$ for all $x \in S$.

Suppose further that $T$ is a nonempty closed subset of $S$ and there exists $B \in \mathbb{N}$ with the following property: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ satisfy $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \text{supp}(h_i)$ for some $i$, and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. Then there
exists \( N \in \mathbb{N} \) such that for all \( \alpha \in \mathbb{N}^n \) with \( \frac{\alpha}{|\alpha|} \in T \), the coefficient of \( X^\alpha \) in \((X_1 + \ldots + X_n)^N p\) is nonnegative.

More precisely, for each \( i \), let \( k(i) \) be the bound from Lemma 1 for \( g_i \) on \( S \), i.e.,

\[
k(i) = \frac{d_i(d_i - 1)}{2} \frac{L(g_i)}{\lambda_i} - d_i,
\]

where \( \lambda_i \) is the minimum of \( g_i \) on \( S \) and \( d_i = \deg g_i \). Then we can take

\[
N \geq \max\{k(g_1), \ldots, k(g_m), B\}.
\]

**Proof.** Given \( \alpha \in \mathbb{N}^n \) with \( \frac{\alpha}{|\alpha|} \in T \). Clearly, it suffices to show that for each \( 1 \leq j \leq m \), the coefficient of \( X^\alpha \) in \((X_1 + \ldots + X_n)^N g_j h_j\) is nonnegative. Suppose \( \beta, \gamma \in \mathbb{N}^n \) are such that \( \beta + \gamma = \alpha \) and the coefficients of \( X^\beta \) in \((X_1 + \cdots + X_n)^N g_j \) and \( X^\gamma \) in \( h_j \) are non-zero. Since \( h_j \in \mathbb{R}[X]^+ \), the coefficient of \( X^\gamma \) in \( h_j \) is positive. Then since we have \( |\beta| > N \geq B \) and \( \alpha = \beta + \gamma \) for \( \gamma \in \text{supp}(h_j) \), \( \frac{\beta}{|\beta|} \in S \) by our assumption. Hence by the choice of \( k(j) \) and Lemma 1, it follows that the coefficient of \( X^\beta \) in \((X_1 + \cdots + X_n)^N g_j \) is nonnegative and we are done.

We now obtain Proposition 1 as a corollary:

**Corollary 3.** Let \( f \in \mathbb{R}[X] \). Suppose that for every \( x \in \Delta_n \) there are \( m \in \mathbb{N} \), homogeneous \( g_1, \ldots, g_m \in \mathbb{R}[X] \), and \( h_1, \ldots, h_m \in \mathbb{R}[X]^+ \) such that

1) \( f = g_1 h_1 + \cdots + g_m h_m \), and
2) \( g_i(x) > 0 \) for \( i = 1, \ldots, m \).

Then there exists \( N \in \mathbb{N} \) such that the coefficients of \((X_1 + \cdots + X_n)^N f\) are nonnegative.

**Proof.** For \( \epsilon > 0 \) and \( x \in \mathbb{R}^n \), let \( B_\epsilon(x) = \{ y \in \mathbb{R}^n \mid ||y - x|| < \epsilon \} \), where \( || \cdot || \) denotes the standard Euclidean norm in \( \mathbb{R}^n \). In other words, \( B_\epsilon(x) \) is the open ball of radius \( \epsilon \) about \( x \). For each \( x \in \Delta_n \), by continuity of the \( g_i \)'s, there is \( \epsilon_x > 0 \) so that a representation of \( f \) as above exists with \( g_i > 0 \) on...
By compactness, we can choose a finite number of $B_{\epsilon_x}(x)$’s covering $\Delta_n$. Then it is enough to show that for each $x \in \Delta_n$ there is an $N_x \in \mathbb{N}$ such that the coefficients of $X^\alpha$ in $(X_1 + \cdots + X_n)^N f$ for $\frac{\alpha}{|\alpha|} \in B_{\epsilon_x}(x)$ are nonnegative. Taking the maximum of the $N_x$’s corresponding to the finite subcover, we are done.

Fix $x \in \Delta_n$, let $M = \max\{\deg(h_i)\}$, and choose $B \geq 2M/\epsilon_x$. Now set $S = \overline{B_{2\epsilon_x}} \cap \Delta_n$ and $T = \overline{B_{\epsilon_x}(x)} \cap \Delta_n$. Then $S$ and $T$ are nonempty and closed and $T \subseteq S$. Hence we need only show that the following property holds: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \text{supp}(h_i)$ for some $i$, and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. We have $|\beta| \geq N \geq B \geq 1$. Thus we have $\frac{2|\gamma|}{|\beta|} \leq \frac{2M}{|\beta|} \leq \epsilon_x$ for $\gamma \in \text{supp}(h_i)$. This gives us

\[
\|x - \beta\| \leq \|\beta - \frac{\alpha}{|\alpha|}\| + \|x - \frac{\alpha}{|\alpha|}\| \\
\leq \|\frac{|\alpha|\beta - |\beta|\alpha}{|\alpha||\beta|}\| + \epsilon_x \\
= \|\frac{|\alpha|\gamma - |\gamma|\alpha}{|\alpha||\beta|}\| + \epsilon_x \\
\leq \|\frac{|\alpha|\gamma}{|\alpha||\beta|}\| + \|\frac{|\gamma|\alpha}{|\alpha||\beta|}\| + \epsilon_x \\
= \frac{2|\alpha||\gamma|}{|\alpha||\beta|} + \epsilon_x \\
= \frac{2|\gamma|}{|\beta|} + \epsilon_x \\
\leq 2\epsilon_x.
\]
Chapter 4

Pólya’s Theorem With Zeros on Vertices

Most of the work in this section can be found in [1]. Recall from Lemma 2 that if \( p \in Po(n,d) \), then \( Z(p) \) must be a union of faces of \( \Delta_n \). In this chapter, we apply Proposition 4 to give a quantitative version of Pólya’s Theorem for forms which are positive on \( \Delta_n \) apart from zeros on the vertices of \( \Delta_n \), i.e., one-dimensional faces. This generalizes the main result from [11].

We begin with some notation. For \( r \in \mathbb{R}, 0 < r < 1 \), and \( i \in \{1, \ldots, n\} \), we define \( \Delta(i,r) \) to be

\[
\{(x_1, \ldots, x_n) \in \Delta_n | \sum_{j \neq i} x_j \leq r\} = \{(x_1, \ldots, x_n) \in \Delta_n | x_i \geq 1 - r\}.
\]

In other words, \( \Delta_n(i, r) \) is the scaled simplex \( r \cdot \Delta_n \) translated by \((1 - r)v_i\) and nestled in the \( v_i \) corner of \( \Delta_n \).

The idea of the proof is to find, for each vertex \( v_i \) where \( p \) has a zero, an \( r \) so that a representation of \( p \) of the type in Proposition 4 exists on \( \Delta_n(i, r) \). Then we apply Lemma 1 to the closure of \( \Delta_n \) minus the corner simplices.

Let \( f \in \mathbb{R}[X] \), say \( f = \sum a_{\alpha} X^\alpha \). Recall that the support of \( f \), denoted \( \text{supp}(f) \), is \( \{\alpha \in \mathbb{N}^n | a_\alpha \neq 0\} \). We define the following measure on the size
CHAPTER 4. PÓLYA’S THEOREM WITH ZEROS ON VERTICES

of the coefficients of $f$:

$$W(f) := \sum_{\alpha \in \text{supp}(f)} |a_{\alpha}|.$$ 

**Lemma 2.** Given homogeneous $f \in \mathbb{R}[X]$ such that

$$f = cX_i^e + \phi(X)$$

for some $i \in \{1, \ldots, n\}$, where $c > 0$ and the degree of $\phi$ in $X_i$ is less than $e$. Let $W = W(f)$ and define

$$r = \frac{c}{c + 2W}, \quad s = \frac{c}{2} \left( \frac{2W}{c + 2W} \right)^e.$$ 

Then $f \geq s$ on $\Delta_n(i, r)$.

**Proof.** Given $x = (x_1, \ldots, x_n) \in \Delta_n$ with $x_i \neq 0$. For $j \neq i$, let $y_j = \frac{x_j}{x_i}$, then, since $\deg f = e$, we have $f(x_1, \ldots, x_n) = x_i^e f(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)$. Let $r$ be as given and suppose $(x_1, \ldots, x_n) \in \Delta_n(i, r)$. Then $x_j \leq r$ for $j \neq i$, and $x_i \geq 1 - r$, hence for each $j \neq i$ we have

$$y_j = \frac{x_j}{x_i} \leq \frac{r}{1 - r} = \left( \frac{c}{c + 2W} \right) \left( \frac{c + 2W}{2W} \right) = \frac{c}{2W}.$$ 

Since the degree of $\phi$ in $X_i$ is less than $e$ and $f$ is homogeneous of degree $e$, every monomial in $\phi(X)$ contains at least one $X_j$ with $j \neq i$, thus $\phi(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)$ has no constant term. Since $\frac{c}{2W} < 1$, it follows that each monomial in $\phi(X)$ evaluated at $(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)$ has absolute value less than $\frac{c}{2W}$. Thus

$$|\phi(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)| \leq \left( \frac{c}{2W} \right)(W) = \frac{c}{2}.$$
and hence
\[
\begin{align*}
f(x_1, \ldots, x_n) &= x_i^c (c + \phi(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)) \\
&\geq (1 - r)^c \left( c - \frac{c}{2} \right) \\
&= \left( \frac{c + 2W - c}{c + 2W} \right)^c \frac{c}{2} \\
&\geq \frac{c}{2} \left( \frac{2W}{c + 2W} \right)^c = s.
\end{align*}
\]

We start with a result for a form \( p \geq 0 \) on \( \Delta_n \) with the set of zeros on \( \Delta_n \) consisting of one vertex.

**Proposition 5.** Given \( p \in P_{n,d}(\Delta_n) \) such that \( Z(p) \cap \Delta_n = \{v_i\} \) for some \( 1 \leq i \leq n \), suppose \( p \) can be written as
\[
p(X) = \sum_{k=1}^{m} M_k \left( c_k X_{i_k}^l + \phi_k(X) \right) + q(X)
\]
where for all \( k \), \( M_k \) is a monomial in \( \{X_1, \ldots, X_n\} \setminus \{X_i\} \), \( c_k > 0 \), the degree in \( X_i \) of \( \phi_k \) is strictly less than \( l_k \), and \( q(X) \) is a polynomial with only non-negative coefficients. Let \( W = W(p) \), \( c = \min\{c_k\} \), \( d = \deg(p) \) and define
\[
r = \frac{c}{c + 2W}, \quad s = \frac{c}{2} \left( \frac{2W}{c + 2W} \right)^d.
\]
Then if
\[
N > \frac{d(d - 1) L(p)}{2 s},
\]
the coefficient of \( X^\theta \) in \( (X_1 + \cdots + X_n)^N p \) is nonnegative for \( \theta/|\theta| \in \Delta_n(i,r) \).

**Proof.** For each \( k \), set \( g_k := c_k X_{i_k}^l + \phi_k(X) \) and apply Lemma 2 to the \( g_k \)'s. Let \( r_k, s_k \) be the bounds for \( g_k \) from Lemma 2, i.e.,
\[
r_k = \frac{c_k}{c_k + 2W(g_k)}, \quad s_k = \frac{c_k}{2} \left( \frac{2W(g_k)}{c_k + 2W(g_k)} \right)^{l_k}.
\]
Then $g_k \geq s_k$ on $\Delta_n(i, r_k)$. Note that the coefficients of the $g_k$'s are a subset of the coefficients of $p$, hence $W(g_k) \leq W(p)$. Thus we have $c \leq c_k \leq W(g_k) \leq W(p)$, and this together with $l_k \leq d$ implies $r \leq r_k$, and $s \leq s_k$.

Since $r \leq r_k$, $\Delta_n(i, r) \subseteq \Delta_n(i, r_k)$ and it follows that, for all $k$, $g_k \geq s$ on $\Delta_n(i, r)$.

We now want to apply Proposition 4 to $p$ with $g_k$ as above, $h_k := M_k$, $S = T = \Delta_n(i, r)$, and $B = 1$. We must check that $S$, $T$, and $B$ satisfy the conditions of Proposition 4. Assume $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\alpha_\beta \in \Delta_n(i, r)$, $\beta + \gamma = \alpha$, $\gamma \in \text{supp}(M_k)$, and $|\beta| \geq 1$. Since $\frac{\alpha}{|\alpha|} \in \Delta_n(i, r)$, we have $\frac{\alpha}{|\alpha|} \geq 1 - r$. Also, $\gamma \in \text{supp}(M_k)$ implies $\gamma_i = 0$, since $M_k$ is a monomial in $\{X_1 \cdots X_n\} \setminus \{X_i\}$, and hence $\alpha_\beta = \beta_\beta$. Since $|\beta| \leq |\alpha|$, it follows that

$$\frac{\beta_\beta}{|\beta|} \geq \frac{\alpha_\beta}{|\alpha|} \geq 1 - r.$$ 

Hence $\frac{\alpha}{|\alpha|} \in \Delta_n(i, r)$ implies $\frac{\beta}{|\beta|} \in \Delta_n(i, r)$. Thus, by Proposition 4, for $\frac{\beta}{|\beta|} \in \Delta_n(i, r)$, the coefficient of $X^\theta$ in $(X_1 + \cdots + X_n)^N p$ is nonnegative. \qed

Let $p \in \mathbb{R}[X]$ and write $p = \sum a_\alpha X^\alpha$, and set:

$$C_0(p) := \min_{\alpha \in \text{supp}(p)} \{|a_\alpha|\}$$

$$C_1(p) := \max_{\alpha \in \text{supp}(p)} \{|a_\alpha|\}$$

Let $d$ be the degree of $p$. We define the following constants for $p$:

$$r(p) := \frac{C_0(p)}{C_1(p) + 2W(p)}, \quad s(p) := \frac{C_0(p)}{2} \left(\frac{2W(p)}{C_1(p) + 2W(p)}\right)^d.$$ 

**Corollary 4.** Suppose $p \in P_{n,d}(\Delta_n)$ such that $Z(p) \cap \Delta_n = \{v_i\}$ for some $1 \leq i \leq n$. Suppose further that for every $\beta \in \Lambda^-_{\Delta_n(p)}$ there is an $\alpha \in \Lambda^+_{\Delta_n(p)}$ with $\alpha \preceq_{\Lambda_{\Delta_n(p)}} \beta$. Then we can find $r, s$ so that for

$$N > \frac{d(d - 1) L(p)}{2},$$
the coefficient of $X^\theta$ in $(X_1 + \cdots + X_n)^N p$ is nonnegative whenever $\frac{\theta}{|\theta|} \in \Delta_n(i, r)$ In particular, we can take $r = r(p)$ and $s = s(p)$.

**Proof.** Suppose in $p$ we have the terms $aX^\alpha$ and $-bX^\beta$ with $\alpha \preceq \{i\} \beta$, where $\beta = (\beta_1, \ldots, \beta_n) \in \Lambda^-(p)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda^+(p)$. This means $\alpha_j \leq \beta_j$ for all $j \neq i$ and $\alpha_j < \beta_j$ for at least one $j$. Since $|\alpha| = |\beta|$, it follows that $\alpha_i > \beta_i$. Then

$$aX_1^{\alpha_1} \cdots X_n^{\alpha_n} - bX_1^{\beta_1} \cdots X_n^{\beta_n} = X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i+1} \cdots X_n^{\alpha_n} (aX_i^{\alpha_i} - bX_i^{\beta_i-\alpha_1} \cdots X_i^{\beta_i} \cdots X_n^{\beta_n-\alpha_n}).$$

Since we can do this for every $\beta \in \Lambda^-$, it is clear we can write $p$ in the form

$$p(X) = \sum_{k=1}^{m} M_k \left( c_k X_i^{l_k} + \phi_k(X) \right) + q(X)$$

where for all $k$, $M_k$ is a monomial in $\{X_1, \ldots, X_n\} \setminus \{X_i\}$, $c_k > 0$, the degree in $X_i$ of $\phi_k$ is strictly less than $l_k$, and $q(X)$ is a polynomial with only non-negative coefficients. Hence we can apply Proposition 5. Let $c$ be the constant from Proposition 5. Note that the $c_k$'s are a subset of the coefficients of $p$, hence $C_0 \leq c$ and $C_1 \geq c$. It follows that

$$r(p) \leq \frac{c}{c + 2W(p)}, \quad s(p) \leq \frac{c}{2} \left( \frac{2W(p)}{c + 2W(p)} \right)^d$$

hence we are done by Proposition 5.  

**Remark 1.** The constants $r$ and $s$ in Corollary 4 are slightly different from the ones found in [1]. This is because we want a “universal” $r$ and $s$, i.e., an $r$ and $s$ that will work for all vertices simultaneously. This will be needed in Chapter 5.

Suppose $p \in P_{n,d}(\Delta_n)$ such that $Z(p) \cap \Delta_n \subseteq \{v_1, \ldots, v_n\}$. Our main result in this chapter is a characterization of when such $p$ is in $Po(n,d)$. 


The idea of the proof is to break up $\Delta_n$ into “corner pieces” $\Delta_n(i, r)$, which contain the zeros, and the rest of $\Delta_n$. Then we apply Proposition 5 to the corner pieces and use Lemma 1 for the remaining piece of $\Delta_n$.

**Theorem 2.** Given $p$ as above, then $p$ is in $\text{Po}(n, d)$ if and only if the following holds: Given $i \in \{1, \ldots, n\}$ with $v_i \in Z(p)$, then for every $\beta \in \Lambda^-(p)$ there is an $\alpha \in \Lambda^+(p)$ with $\alpha \preceq \{i\} \beta$.

Moreover, if $p \in \text{Po}(n, d)$, then we can take

$$N > \max \left\{ \frac{d(d - 1) L(p)}{2s}, \frac{d(d - 1) L(p)}{2\lambda} \right\}$$

where $r = r(p)$, $s = s(p)$, $\lambda$ is the minimum of $p$ on $\Delta_n \setminus \left( \bigcup_{v_i \in Z(p)} \Delta_n(i, r) \right)$, and $d$ is the degree of $p$.

**Proof.** If $p$ is in $\text{Po}(n, d)$, the given condition holds by Proposition 3.

Now suppose that the assumption holds. Then by Corollary 4, if $\frac{\theta}{|\theta|} \in \Delta_n(i, r)$ for some $i$, then the coefficient of $X^\theta$ in $(X_1 + \cdots + X_n)^N p$ is non-negative. Note that at each vertex $v_i \in Z(p)$, the bound on $N$ is the same. If $\frac{\theta}{|\theta|} \notin \Delta_n(i, r)$, for all $v_i \in Z(p)$, then $\frac{\theta}{|\theta|} \in \Delta_n \setminus \left( \bigcup_{v_i \in Z(p)} \Delta_n(i, r) \right)$ and the coefficient of $X^\theta$ is nonnegative by Lemma 1.

We obtain as a corollary a new proof of the main result in [11]. As in [11], we say that a form $p$ of degree $d$ which is nonnegative on $\Delta_n$ has a simple zero at $v_j$ if the coefficient of $X_j^d$ in $p$ is zero, but the coefficient of $X_j^{d-1}X_i$ is non-zero (and necessarily positive) for each $i \neq j$.

**Corollary 5.** Suppose $p$ is positive on $\Delta_n$ except for simple zeros at some $v_j$’s. Then $p \in \text{Po}(n, d)$ and there is a bound for the exponent $N$ as in Theorem 2.

**Proof.** Given $j$ so that $v_j \in Z(p)$, then $p$ has no terms of the form $aX_j^d$ and necessarily has terms of the form $aX_j^{d-1}X_i$ for all $i \neq j$. Suppose $\beta \in \Lambda^-(p)$
and fix \( v_j \in Z(p) \). Pick \( i \neq j \) so that \( \beta_i \neq 0 \) and let \( \alpha \) be the exponent of the \( X_j^{d-1}X_i \) term. Then \( \alpha \in \Lambda^+(p) \) and \( \alpha \preceq_{\{j\}} \beta \). Hence, we are done by Theorem 2.

\[ \square \]

**Example 3.** This example is from [11], using the bound from Theorem 2. For \( 0 < \alpha < 1 \), let

\[ p(x, y, z) := x(y - z)^2 + y(x - z)^2 + z(x - y)^2 + \alpha xyz. \]

Then \( p \in P_{3,3}(\Delta_3) \) with zeros at all three vertices.

We start by computing the bound on the corners. In this case, we have \( d = 3, \ C_0(p) = \alpha, \ C_1(p) = 2, \) and \( W(p) = 12 - \alpha \), hence the constants from Corollary 4 are

\[ r(p) = \frac{\alpha}{26 - 2\alpha}, \quad s(p) = \frac{\alpha}{2} \left( \frac{24 - 2\alpha}{26 - 2\alpha} \right)^3, \quad L(p) = \max \left\{ 1, \frac{6 - \alpha}{6} \right\} = 1. \]

Thus

\[ \frac{d(d - 1) L(p)}{2} s = \frac{6}{\alpha} \left( \frac{26 - 2\alpha}{24 - 2\alpha} \right)^3. \]

From calculations in [11], we have that the minimum of \( p \) on the interior of the closure of \( \Delta_3 \) minus the three corners is \( \frac{\alpha}{27} \). Thus,

\[ \frac{d(d - 1) L(p)}{2} \lambda = \frac{81}{\alpha}. \]

Putting this together, if

\[ N > \max \left\{ \frac{6}{\alpha} \left( \frac{26 - 2\alpha}{24 - 2\alpha} \right)^3, \frac{81}{\alpha} \right\}, \]

then \( (X_1 + \cdots + X_n)^N p \) has nonnegative coefficients. As \( \alpha \) approaches 0, this \( N \) behaves like \( \frac{1}{\alpha} \). From [11] we have \( (X_1 + \cdots + X_n)^N p \) has nonnegative coefficients for \( N \geq \frac{18}{\alpha} - 3 \), and this bound is sharp if \( \frac{18}{\alpha} - 3 \in \mathbb{N} \). Hence, the computed bound has the same order of growth as the true bound, \( \frac{18}{\alpha} - 3 \).
Chapter 5

Zeros On A Two Dimensional Face

In this chapter, we look at forms \( p \in P_{n,d}(\Delta_n) \) with \( Z(p) \cap \Delta_n \) a union of two-dimensional faces. We show that in this case, if \( p \) satisfies the condition of Proposition 3 with \( \alpha \preceq \beta \) replaced with \( \alpha \prec \beta \), then \( p \in Po(n,d) \).

**Definition 3.** For \( r \in \mathbb{R} \), \( 0 < r < 1 \) and \( I \subseteq \{1, \ldots, n\} \), let \( \Delta(I, r) \) be the set 

\[
\{(x_1, \ldots, x_n) \in \Delta_n \mid \sum_{j \notin I} x_j \leq r \} = \{(x_1, \ldots, x_n) \in \Delta_n \mid \sum_{i \in I} x_i \geq 1 - r \}.
\]

We start with the case where \( Z(p) \cap \Delta_n \) consists of one two-dimensional face. For ease of exposition, assume \( Z(p) \cap \Delta_n = F(I) \), where \( I = \{1, 2\} \). Given \( r \in (0, \frac{1}{2}) \) and \( t \in (0, \frac{r}{2}] \), define

\[
U(r, t) = \{(x_1, \ldots, x_n) \in \Delta_n \mid x_1, x_2 \geq r - t, \text{ and } \sum_{i=3}^{n} x_i < t \} \subseteq \Delta(I, t)
\]

It is easy to see that \( \Delta(I, t) \subseteq U(r, t) \cup \Delta(1, r) \cup \Delta(2, r) \).

To prove our main theorem, we are going to find \( r \) and \( t \) so that we can apply Proposition 4 to \( \Delta(1, r) \), \( \Delta(2, r) \), and \( U(r, t) \), and then apply Lemma 1 to \( \Delta_n \setminus \Delta(I, t) \).
CHAPTER 5. ZEROS ON A TWO DIMENSIONAL FACE

Given a form \( p \), let \( d \) be the degree of \( p \). Recall that we define

\[
    r(p) := \frac{C_0(p)}{C_1(p) + 2W(p)}, \quad s(p) := \frac{C_0(p)}{2} \left( \frac{2W(p)}{C_1(p) + 2W(p)} \right)^d,
\]

and set

\[
    t(p) = r(p) \left( \frac{C_0(p)}{C_1(p) + 2W(p)} \right), \quad u(p) = \frac{C_0(p)}{2} \left( \frac{2C_0(p)W(p)}{(C_1(p) + 2W(p))^2} \right)^d.
\]

**Proposition 6.** Let \( \tilde{X} = (X_3, \ldots, X_n) \), so that for \( \Gamma = (\gamma_3, \ldots, \gamma_n) \in \mathbb{N}^{n-2} \), \( \tilde{X}^{\Gamma} \) denotes \( X_3^{\gamma_3} \cdots X_n^{\gamma_n} \). Suppose \( p \in \mathbb{R}[X] \) is a form which can be written as

\[
    \sum_{j=1}^{m} \tilde{X}^{\Gamma_j} \left( c_j X_1^{k_j} X_2^{l_j} + \phi_j(X) \right) + q(X)
\]

where for all \( j \), \( c_j > 0 \) and the degree in \( X_1 X_2 \) of \( \phi_j \) is strictly less than \( k_j + l_j \), and \( q(X) \) is a form with only non-negative coefficients. Let \( W = W(p), r = r(p), t = t(p), \) and \( u = u(p) \). Then for \( \theta \in \mathbb{N}^n \) with \( \frac{\theta}{|\theta|} \in U(r, t) \) the coefficient of \( X^\theta \) in \( (X_1 + \cdots + X_n)^N p \) is non-negative for

\[
    N > \frac{d(d-1)L(p)}{2u}.
\]

**Proof.** For each \( j \), set \( g_j := c_j X_1^{k_j} X_2^{l_j} + \phi_j(X) \) and \( h_j := \tilde{X}^{\Gamma_j} \). We are going to apply Proposition 4 with \( S = T = U(r, t) \), thus we need a lower bound for the \( g_j \)'s on \( U(r, t) \). Given \( x = (x_1, \ldots, x_n) \in U(r, t) \), then \( x_1, x_2 \geq r - t \) and \( x_i \leq t \) for \( i = 3, \ldots, n \). Fix \( 1 \leq j \leq n \) and let \( X^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n} \) be a monomial in \( \phi_j(X) \). Then

\[
    \frac{x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}}{x_1^{k_j} x_2^{l_j}} = \frac{x_3^{\beta_3} \cdots x_n^{\beta_n}}{x_1^{k_j-l_j} x_2^{l_j}} \leq \frac{t^{\beta_3+\cdots+\beta_n}}{(r-t)^{(k_j+l_j)-(\beta_1+\beta_2)}}.
\]

Since \( g_j \) is a form of degree \( k_j + l_j \), \( (k_j + l_j) - (\beta_1 + \beta_2) = \beta_3 + \cdots + \beta_n \), hence

\[
    \frac{t^{\beta_3+\cdots+\beta_n}}{(r-t)^{(k_j+l_j)-(\beta_1+\beta_2)}} = \frac{t^{\beta_3+\cdots+\beta_n}}{(r-t)^{\beta_3+\cdots+\beta_n}} = \left( \frac{t}{r-t} \right)^{\beta_3+\cdots+\beta_n}.
\]
Since \( \frac{t}{r-t} < 1 \), we have \( \left( \frac{t}{r-t} \right)^{\beta_3 + \cdots + \beta_n} \) < \( \left( \frac{t}{r-t} \right) \). Hence for each term 

\[
M(X) = b_{\beta_3} X_1^{\beta_1} X_2^{\beta_2} \cdots X_n^{\beta_n}
\]

in \( \phi_j(X) \),

\[
\left| \frac{M(x_1, x_2, \ldots, x_n)}{x_1^{k_j} x_2^{l_j}} \right| \leq |b_{\beta_j}| \left( \frac{t}{r-t} \right).
\]

Note that

\[
r - t = r \left( 1 - \frac{C_0}{C_1 + 2W} \right) = r \left( \frac{C_1 - C_0 + 2W}{C_1 + 2W} \right)
\]

and thus,

\[
|\phi(x_1, \ldots, x_n)/x_1^{k_j} x_2^{l_j}| \leq \left( \frac{t}{r-t} \right) W
\]

\[
= \left( \frac{C_0}{C_1 + 2W} \right) \left( \frac{C_1 + 2W}{C_1 - C_0 + 2W} \right) W
\]

\[
= \left( \frac{C_0}{C_1 - C_0 + 2W} \right) W
\]

\[
\leq \left( \frac{C_0}{2W} \right) W = \frac{C_0}{2}.
\]

Since \( x_1, x_2 \geq r - t \), and noting that \( k_j + l_j \leq d \) and \( C_1 \geq C_0 \),

\[
x_1^{k_j} x_2^{l_j} \geq (r-t)^{k_j+l_j}
\]

\[
= \left[ r \left( \frac{C_1 - C_0 + 2W}{C_1 + 2W} \right) \right]^{k_j+l_j}
\]

\[
\geq \left[ r \left( \frac{2W}{C_1 + 2W} \right) \right]^d
\]

\[
= \left[ \left( \frac{C_0}{C_1 + 2W} \right) \left( \frac{2W}{C_1 + 2W} \right) \right]^d
\]

\[
= \left( \frac{2C_0 W}{(C_1 + 2W)^2} \right)^d.
\]
Therefore we obtain the bound on $N$ as given.
Theorem 3. Given \( p \in P_{n,d}(\Delta_n) \) with \( Z(p) \cap \Delta_n = F(I) \). Suppose for every \( \beta \in \Lambda^-(p) \) there are \( \alpha, \gamma, \delta \in \Lambda^+(p) \) such that \( \alpha \prec_{\{1\}} \beta, \gamma \prec_{\{2\}} \beta, \delta \prec_{\{1,2\}} \beta \). Then \( p \in Po(n,d) \). In particular, if we let \( r = r(p), s = s(p), t = t(p), u = u(p) \), and let \( \lambda \) be the minimum of \( p \) on \( \Delta_n \setminus \Delta(I,t) \), then whenever

\[
N > \max \left\{ \frac{d(d-1)L(p)}{2s}, \frac{d(d-1)L(p)}{2u}, \frac{d(d-1)L(p)}{2\lambda} \right\}
\]

the coefficients of \( (X_1 + \ldots + X_n)^N p \) are nonnegative.

Proof. We are going to apply Proposition 4 to appropriate closed subsets of the simplex. In particular, we use \( \Delta(1,r), \Delta(2,r), \) and \( U(r,t) \), and recall that \( \Delta(I,t) \subseteq U(r,t) \cup \Delta(1,r) \cup \Delta(2,r) \). We then will apply Lemma 1 to \( \Delta_n \setminus \Delta(I,t) \).

Let \( N \) be as in the statement. Given \( \theta \in \mathbb{N}^n \) with \( |\theta| = N + d \) consider the \( X^\theta \) term in \( (X_1 + \cdots + X_n)^N p \). If \( \frac{\theta}{|\theta|} \in \Delta(1,r) \cup \Delta(2,r) \), the coefficient of \( X^\theta \) is nonnegative, by Corollary 4.

Suppose \( \frac{\theta}{|\theta|} \in U(r,t) \). By assumption, for any \( \beta \in \Lambda^- \) there exists an \( \delta \in \Lambda^+ \) such that \( \delta \prec_{\{1,2\}} \beta \). As in the vertex case, we can write

\[
p = \left( \sum_{j=1}^{m} X^\Gamma_j \left( c_j X_1^{k_j} X_2^{l_j} + \phi_j(X) \right) \right) + q(X),
\]

where the degree in \( X_1X_2 \) of \( \phi_j \) is strictly less than \( k_j + l_j \), and \( q(X) \) is a polynomial with only non-negative coefficients. Then by Proposition 6, the coefficient of \( X^\theta \) is nonnegative.

Finally, we note that \( p > 0 \) on \( \Delta_n \setminus \Delta(I,t) \) and hence we can apply Lemma 1 in the case where \( \frac{\theta}{|\theta|} \in \Delta_n \setminus \Delta(I,t) \). Therefore \( p \in Po(n,d) \) with the bound on \( N \) as given.

\[ \square \]

Remark 2. Theorem 3 with a more complicated bound also follows from [8].
Example 4. Consider the following family of psd polynomials, defined in [12]: For \( n \in \mathbb{N} \), and \( 0 < \epsilon < 1 \), let

\[
M_n(x_1, \ldots, x_n) := x_1^2 \cdots x_{n-1}^2 \left( \sum_{i=1}^{n-1} x_i^2 \right) + x_n^{2n} - (n - \epsilon) x_1^2 \cdots x_n^2.
\]

Since \( M_n \) is psd, it is nonnegative on \( \Delta_n \). It is easy to see \( Z(M_n) = \bigcup_{1 \leq i \leq j \leq n-1} F(\{i, j\}) \).

Let \( \beta = (2, \ldots, 2) \), then \( \Lambda^{-}(M_n) = \{\beta\} \). Given \( I = \{i, j\} \) with \( 1 \leq i \leq j \leq n - 1 \), let \( \alpha \) be the exponent of \( X_1^2 \cdots X_{n-1}^2 \cdot X_i^2 \in \Lambda^{+}(M_n) \). Then clearly, \( \alpha \prec_{(i,j)} \beta \) and \( \alpha \prec_{(i)} \beta \). Thus the conditions of Theorem 3 hold and \( M_n \in Po(n, d) \).
Bibliography


